Adaptive Mesh Refinement for Multiscale FEM for the Eddy Current Problem in Laminated Materials

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Solving the eddy current problem on a domain consisting of many thin laminates using a finite element approach requires a mesh with a high number of elements to resolve each laminate, which in turn results in a computationally unfeasible amount of degrees of freedom in the equation system. This work is based on a known multiscale ansatz which mitigates this problem by solving an averaged problem on a coarse mesh which does not resolve the single laminates and superposing the solution with adequately chosen ansatz functions. This known method is enhanced by the development of a local error estimator based on flux equilibration which allows for adaptive mesh refinement. The ideas will be presented for the time-harmonic single component current vector potential (SCCVP) in two dimensions.

***Index Terms*—eddy currents, error estimation, multiscale**

I. Problem Setting

CONSIDER A DOMAIN Ω consisting of a laminated subdomain Ω_m and a surrounding domain Ω_0, see Fig. 1. The width of one laminate is d_1 and the gaps between the laminates are of width d_2. The total period width in Ω_m will be denoted $d = d_1 + d_2$. In the applications the laminates represent iron sheets separated by air gaps, surrounded by air.

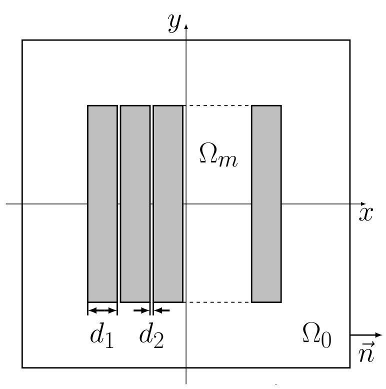

Fig. 1. Schematic of the domain $\Omega = \Omega_m \cup \Omega_0$.

The problem formulation for the SCCVP in two dimensions is derived from the assumption of independence of the solution from the z coordinate, i.e. $T = (0,\ 0,\ u(x, y))^T$, corresponding to the magnetic field intensity. In the weak form it is given as: Find $u \in H^1(\Omega),\ u = \alpha$ on $\partial\Omega$ with a given Dirichlet boundary condition α so that

$$\int \rho \nabla u \nabla v + i\omega\mu uv \ d\Omega = 0 \tag{1}$$

holds for every $v \in H^1(\Omega),\ v = 0$ on $\partial\Omega$, where ρ corresponds to the electric resistivity, μ to the magnetic permeability and ω to the angular frequency.

II. The Multisscale Ansatz

Following the techniques described in [1], the solution of (1) is approximated using the ansatz

$$u = u_0 + \phi(x)u_1, \tag{2}$$

where the functions u_0 and u_1 are defined on a coarse mesh which does not resolve the single laminates. The micro-shape function ϕ is equal to 0 in air and a quadratic polynomial in each laminate.

The ansatz (2) is used in (1) for the trial function and the test function. This results in the coupled system: Find $u_0 \in H^1(\Omega),\ u_0 = \alpha$ on $\partial\Omega$ and $u_1 \in H^1(\Omega_M),\ u_1 = 0$ on the top and bottom boundary of Ω_M, so that

$$\begin{aligned} \int & \overline{\rho}\nabla u_0 \nabla v_0 + \overline{\rho\phi}(\nabla u_0 \nabla v_1 + \nabla u_1 \nabla v_0) + \\ & \overline{\rho\phi_x^2} u_1 v_1 + \overline{\rho\phi^2}\nabla u_1 \nabla v_1 + i\omega\overline{\mu} u_0 v_0 + \\ & i\omega\overline{\mu\phi^2}(u_0 v_1 + u_1 v_0) + i\omega\overline{\mu\phi^2} u_1 v_1 \ d\Omega = 0 \end{aligned} \tag{3}$$

holds for every $v_0 \in H^1(\Omega),\ v_0 = 0$ on $\partial\Omega$ and every $v_1 \in H^1(\Omega_M),\ v_1 = 0$ on the top and bottom boundary of Ω_M, where a bar indicates that the respective quantity is averaged over one period and ϕ_x is the derivative of ϕ with respect to the x coordinate.

III. The Error Estimator

A. The Theorem of Prager and Synge

The error estimator is based on the theorem of Prager and Synge, which states that if u is the solution of $-\kappa\Delta u = f$ and $\sigma \in H(\mathrm{div})$ a solution of $\mathrm{div}\,\sigma + f = 0$ satisfying homogenous Neumann boundary conditions, then for every $v \in H^1$ satisfying the same Dirichlet boundary conditions as u, there holds

$$\|(\nabla u - \nabla v)\|_\kappa^2 + \|(\nabla u - \kappa^{-1}\sigma)\|_\kappa^2 = \|(\nabla v - \kappa^{-1}\sigma)\|_\kappa^2 \tag{4}$$

with the energy norm $\|.\|_\kappa$ defined as $\|u\|_\kappa^2 = \int \kappa \nabla u \overline{\nabla u}\ d\Omega$.

The idea is to substitute the FEM solution u_h for v in (4) to get an upper bound for the error in the energy norm:

$$\|\nabla u - \nabla u_h\|^2_\kappa \leq \|\nabla u_h - \kappa^{-1}\sigma\|^2_\kappa. \quad (5)$$

In [2] it is described how to construct a suitable function σ efficiently by calculating local correctors to the numerical flux $\kappa\nabla u_h$. Using this σ, the right hand side of (5) can be evaluated directly.

B. Application to the Multiscale Problem

Consider (3) only for test functions $v_1 = 0$:

$$\begin{aligned} \int \overline{\rho}\nabla u_0 \nabla v_0 + \overline{\rho\phi}\nabla u_1 \nabla v_0 + \\ i\omega\overline{\mu}u_0 v_0 + i\omega\overline{\mu\phi^2}u_1 v_0 \; d\Omega = 0 \end{aligned} \quad (6)$$

If only u_0 is considered an unknown, depending on a given function u_1, the expression (6) can be split into a bilinear form on the FEM space of u_0 and a linear right hand side. Using Green's formula in the term containing $\nabla u_1 \nabla v_0$ leads to

$$\begin{aligned} &\int \overline{\rho}\nabla u_0 \nabla v_0 + i\omega\overline{\mu}u_0 v_0 \; d\Omega = \\ &\int \overline{\rho\phi}\Delta u_1 v_0 - i\omega\overline{\mu\phi^2}u_1 v_0 \; d\Omega - \sum_T \int_{\partial T} \overline{\rho\phi}\frac{\mathrm{d}u_1}{\mathrm{d}n} v_0 \; ds \end{aligned} \quad (7)$$

where the sum iterates over all finite elements.

The error estimator can be naturally extended to allow for the right hand side f to contain normal jump terms. To account for the mass term, the setting of the theorem of Prager and Synge has to be extended. The following relation can be shown:

Let u be the solution of $-\kappa\Delta u + \gamma u = f$, then for every $v \in H^1$ satisfying the same Dirichlet boundary conditions as u and every solution $\sigma \in H(\mathrm{div})$ of $\mathrm{div}\,\sigma + f - \gamma v = 0$ satisfying homogenous Neumann boundary conditions, the estimation

$$\|\nabla u - \nabla v\|^2_\kappa + \|u - v\|^2_\gamma \leq \|(\nabla v - \kappa^{-1}\sigma)\|^2_\kappa. \quad (8)$$

holds.

The construction of the estimator can be modified to work in this extended setting, again allowing for the construction of a reliable local error estimator.

Similarly, u_1 fulfills

$$\begin{aligned} &\int \overline{\rho\phi^2}\nabla u_1 \nabla v_1 + (\overline{\rho\phi_x^2} + i\omega\overline{\mu\phi^2})u_1 v_1 \; d\Omega = \\ &\overline{\rho\phi}\Delta u_0 v_1 - \overline{\rho\phi}[\nabla u_0 \cdot n]v_1 - i\omega\overline{\mu\phi^2}u_0 v_1 \; d\Omega \end{aligned} \quad (9)$$

for all suitable test functions v_1, which is in the same setting as the corresponding equation for u_0.

IV. Numerical Example

For a simple numerical example a setting with 10 laminates is chosen. The domain Ω_m is given as a square with a side width of 2mm. The fill factor is given as 0.9, i.e. $d_1 = 0.9d$. The electric resistivity ρ is given as $\rho = 5 \cdot 10^{-7}\ \Omega m$ in iron and, to ensure convergence of the numerical method, $\rho = 5\ \Omega m$ in air. The magnetic permeability is $\mu = \mu_0$ in air and $\mu = 10000\mu_0$ in iron. The frequency is 50Hz.

Fig. 2 shows a comparison of the curl of the reference solution, i.e. the current density, and the curl of the multiscale solution. For the sake of visibility, only the top left corner of Ω_m is shown. A fairly good agreement between the two solutions can be observed. Note that the ansatz (2) introduces a small additional error, mainly in the corners of the sheets, which could be improved by choosing a higher order approach.

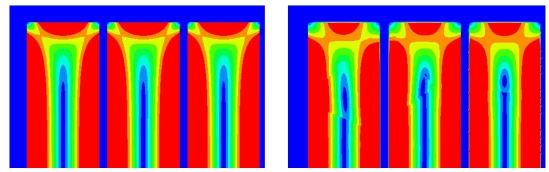

Fig. 2. Absolute value of $\mathrm{curl}\, u$ for the reference solution (left) and the multiscale solution (right).

As a next step the error estimator developed above was used to adaptively refine the mesh to improve the quality of the multiscale solution. Fig. 3 compares the performance of this approach with refining the mesh uniformly in each step. It can be seen that the adaptive refinement gives clearly better results for the same number of unknowns in the equation system.

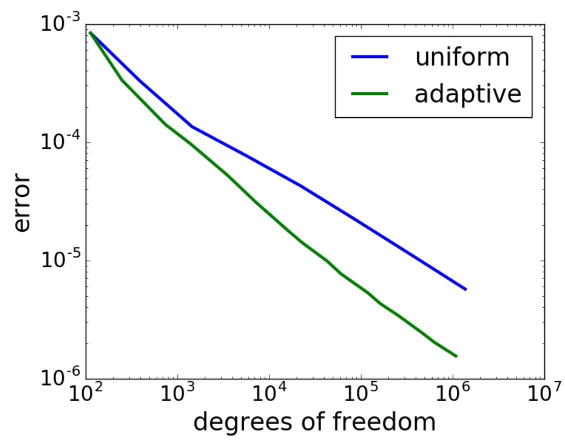

Fig. 3. Error of u_0 and u_1 depending on the degrees of freedom.

V. Conclusion

A local error estimator allowing for adaptive mesh refinement has been successfully developed for the case of the SCCVP. Since in [2] a similar error estimator for functions in $H(\mathrm{curl})$ has been presented using an analogue to the theorem of Prager and Synge for the magnetostatic setting, the ideas presented here might be extendable to multiscale formulations for the magnetic vector potential, which is a topic of future work.

VI. Acknowledgment

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References

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